ON THE THEORY OF CALCULATION OF SERVICE LIFE OF A STRUCTURE

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ABSTRACT

This work throws light on some problems in the theory of the determination of the permissible service life of a structure subject to fatigue with the calculation of a threshold valve for the fatigue life. Formulas have been obtained for logarithmic normal distribution for the calculation of lower limits of error in the estimation of a minimum life obtained from fatigue tests. A method based on the principle of the greatest probability has been worked out for the determination of the minimum value. The reliability and effectiveness of the latter is discussed. For the evaluation of an allowable service life a general safety criterion is proposed which with sample improvements permits a standardization of safety factors in the determination of the service life.

INTRODUCTION

The basic phenomenon of fatigue which must be taken into account in the evaluation of the permissible service life of a structure consists of a scatter of the fatigue life caused by great sensitivity to fatigue in various types of structures and not least to various technological factors. The number of factors that have an influence is so large that it is impossible to consider the dependence of the length of life upon varying stresses in any other way than as a random value with certain statistical distribution whose parameters must be made the subject of an estimation in the determination of the permissible service life.

Results from the statistical research of recent years have established the character of the distribution and also the magnitude of the scatter of the characteristics of the fatigue endurance. Some analytic expressions have been put forward for the distribution of the number of loading cycles until the point when failure in the structure occurs. The most acceptable would appear to be the proposals of Weibull [1], Freudenthal and Gumbel [2], and Sörensen [3]. The most convenient for theoretical application would appear to be a logarithmically normal distribution [3].

The magnitude of the scatter, characterized by the standard deviation departure of the logarithms of the number of load cycles leading to failure, $\sigma \ln N$, is affected both by the type of structure and quality of manufacture and by the level of the fatigue loads. Figure 1 shows, from experiment, the standard deviation of the natural logarithms $\bar{S}_{\ln N}$ of the failure life as a function of the mean value of life $\bar{N}$.

$$\bar{S}_{\ln N} = \frac{1}{n-1}\sum_{i=1}^{n} \ln N_i - \overline{\ln N}^2$$

$$\overline{\ln N} = \frac{1}{n}\sum_{i=1}^{n} \ln N_i \quad \bar{N} = e^{\overline{\ln N}}$$

The relationship is based upon the works referred to [2,4–5], which deal with the fatigue endurance for plain and notched test specimens of aluminum alloys and steel and also actual structures with limited numbers of weak points. It will be seen from Fig. 1 that the scatter increases considerably with an increase in the mean value of the length of life to which regard must be paid in the determination of the factor of safety of the fatigue strength. In recommendations now in existence for the estimation of the service life of structures [6,7] increase in scatter is not taken into account in the standardization of the x factors of safety (with reduced level of stress).

In addition to the increase in scatter with a reduction in the fatigue loads occurring, yet another phenomenon is observed which demands a special method of treatment in the evaluation of a maximum allowable service life. This phenomenon of the probability distribution of the fatigue strength consists of a departure from the logarithmic normal within the region of frequency of loading which corresponds to a small probability of failure. This departure is evidently caused by the existence of a definite number of cycles N_0 up to which the probability of failure is nil. Following Freudenthal [8] we can call the quantity N_0 "minimum life in fatigue." A characteristic form for the dependence of the probability of failure upon

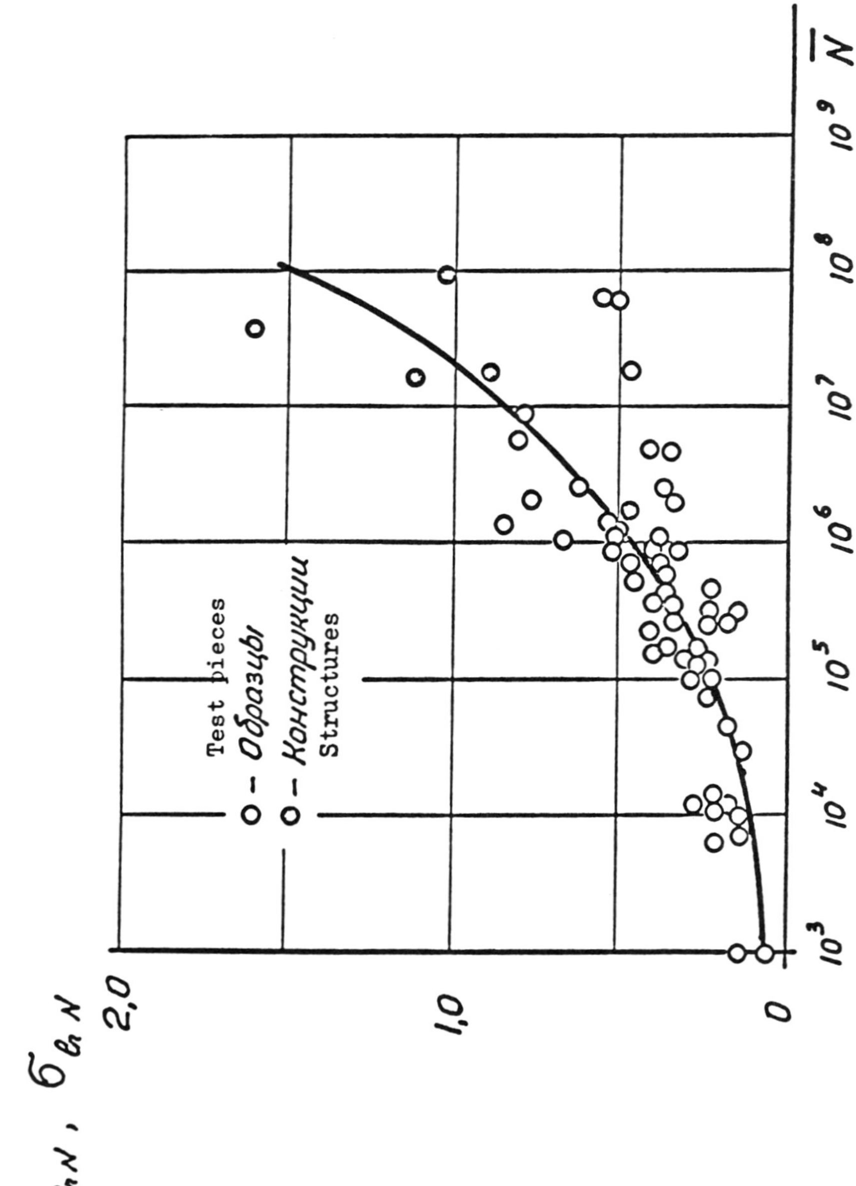

Figure 1. Dependence of standard deviation (S_{nN}) on magnitude of (geometrical) mean value $\bar{N}$.

the number of loading periods, i.e., the distribution function of the length of life until failure, has been taken from Ref. 3 and is shown in Fig. 2. As is shown in this reference, one can assume that the log normal distribution depends upon the difference between the number of cycles to failure and the minimum life $N - N_0$.

Investigations in accordance with Refs. 2, 4, 5, and 9 make it possible in a first approximation to determine a value of N_0 related to the mean life of the test specimens (Fig. 3). According to M. N. Stepnikov [9] this relationship can be written

$$\ln N_0 = 0.86\, \overline{\ln N}$$

The existence of a minimum life can markedly alter the length of the allowable service life compared with the zero-minimum life.

Nowadays the estimation of the allowable service life is carried out as a rule by two different methods which are based on the theory of the random log-normal distribution of the length of life until failure with a zero-minimum life. As is known, it is necessary and sufficient for clearly establishing the probability of failure after a log-normal distribution to know the actual mean value (mathematical expectancy) $a_{\ln N}$ and the standard deviation $\sigma_{\ln N}$ for the logarithms of the length of life until failure.* In one method [see, for example, Ref. (10)] both the quantities $a_{\ln N}$ and $\sigma_{\ln N}$ are determined from tests carried out on the fatigue endurance of actual structural elements. The permissible number of loading periods for a given probability of failure for the structure P_{norm} and the maximum permissible probability of error in the determination of the service life $P_{\text{error}} = \gamma$ is calculated in accordance with the formula

$$\ln N_{\text{allowable}} = \overline{\ln N} - K_p \overline{S}_{\ln N}$$

The quantity K_p [see Ref. (10)] is determined by the condition that the probability will be such that the quantity p will be less than the given P_{norm}, i.e., that the probability of error for a given service life will not be greater than γ. If the number of test specimens in the fatigue tests is sufficiently great, $\overline{\ln N} \to a_{\ln N}$, $\overline{S_{\ln N}} \to \sigma_{\ln N}$, $aK_p \to U_p$, where U_p is the P-quantile for standardized normal population. In such a case

$$\ln N_{\text{allowable}} = a_{\ln N} + U_p \sigma_{\ln N}$$

Where precise information is available about the magnitude of the minimum life

$$\ln N_{\text{allowable}} = N_0$$

* For the sake of simplicity natural logarithms are used here.

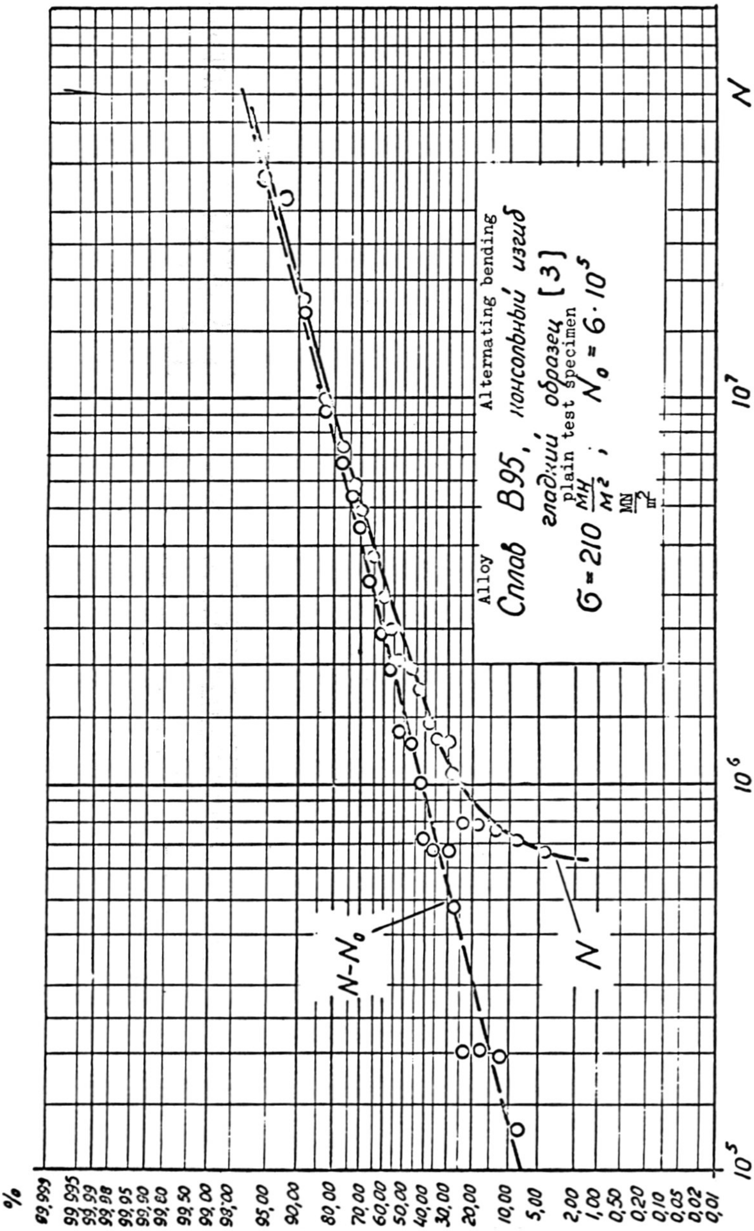

Figure 2. Distribution of fatigue endurance in the neighborhood of the minimum life.

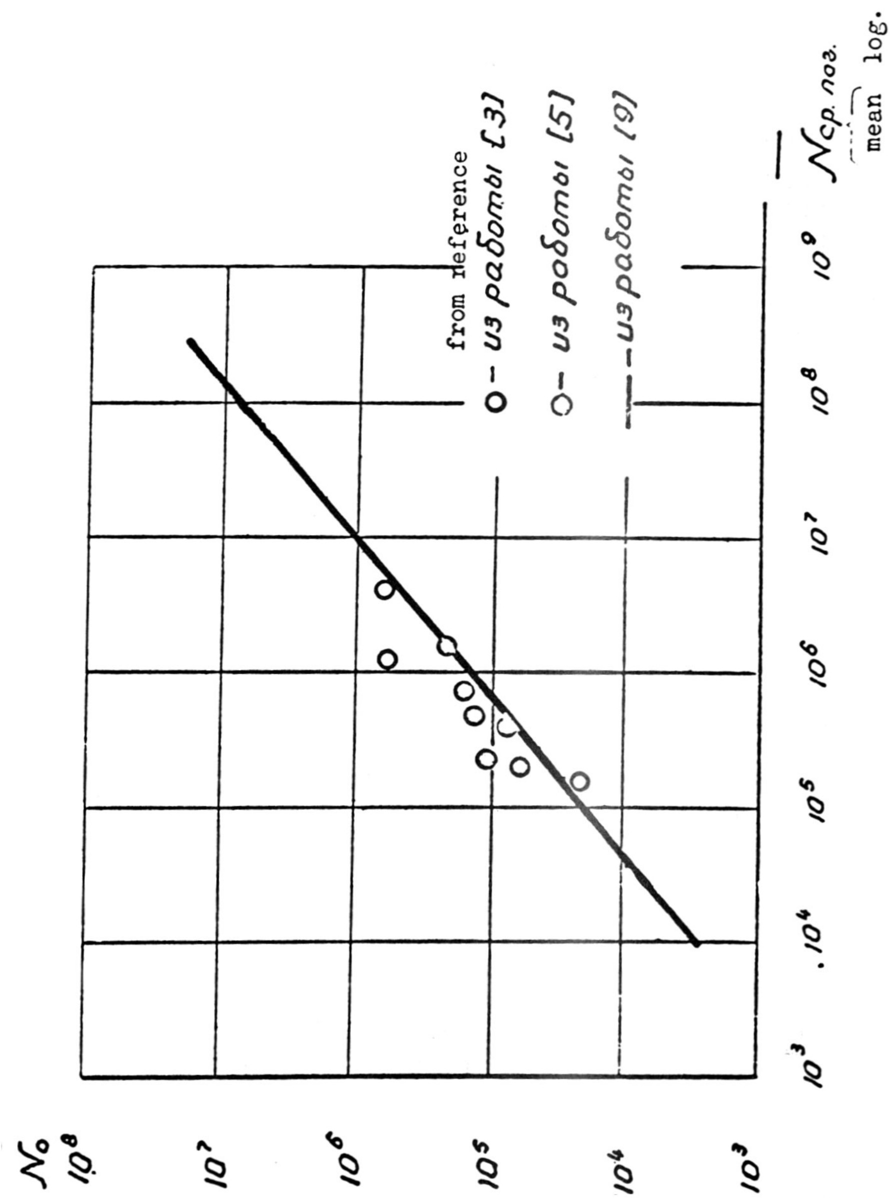

Figure 3. Dependence of minimum life on mean endurance.

Figure 4 shows a comparison between allowable service lives when $N_0 = 0$ and when $N_0 \neq 0$ for the relationship $N_0 = f(\bar{N})$ and $\sigma_{\ln N} = f(\bar{N})$ given in full lines in Figs. 1 and 3. The number of test specimens used in the tests has been assumed to be infinitely large. From the comparison quoted it follows that at low stresses a neglected determination of the minimum life, particularly with a large machine park with consequently greater probability that failure will not occur, can reduce the allowable service life of the structure to a tenth or less. On the other hand, for relatively high alternating stresses which determine the service life of the structure a neglected determination of the minimum life can lead to an unmotivated increase in the service life.

The other method, which has been adopted in the standardization of safety factors in a number of strength and flight safety standards, is based on the magnitude of the standard deviation $\sigma_{\ln N}$ of the fatigue endurance. In the majority of cases a relatively low value of the quantity σ_{norm} is used which agrees with tests on built-up structures with relatively high levels of fatigue stresses. The British airworthiness requirements [6] correspond approximately to the magnitude $\sigma_{\text{norm}} = 0.35$ at $P_{\text{norm}} = 0.0001$ and $P_{\text{error}} = 0.1$. The reliability of this method of evaluating the service life depends principally on the actual values for the parameter $\sigma_{\ln N}$ and keeps to the allowable limits for $N_0 = 0$ only when $\sigma_{\ln N} < 0.35$.

The existence of a minimum life in fatigue can extend considerably the range of the parameters over which sufficient safety is still retained in the calculation of the service life. In Fig. 5 the dotted parts of parameters σ and $\Upsilon/\chi\ e^a/N_0$ indicate the region within which the given safety is attained, i.e., $P_{\text{failure}} < P_{\text{norm}}$ and $P_{\text{error}} < \gamma$. In the case given, the parameters a and σ represent the expectancy and the standard deviation of the quantity $\ln (N - N_0)$. The regions have been designed for the case where the calculation of the service life is based on the mean average value of the length of life (British Requirements):

$$N_{\text{allowable}} = \frac{\bar{N}}{\eta_1}$$

and on the choice of the minimum value of the length of life

$$N_{\text{allowable}} = \frac{N_{\min}}{\eta_2}$$

$N_{\min}$ is the least number of cycles to failure in fatigue testing. η_1 and η_2 are required factors of safety.

As will be seen from Fig. 5, a satisfactory safety is secured only for very large values of the minimum life. In order that this method of evaluation

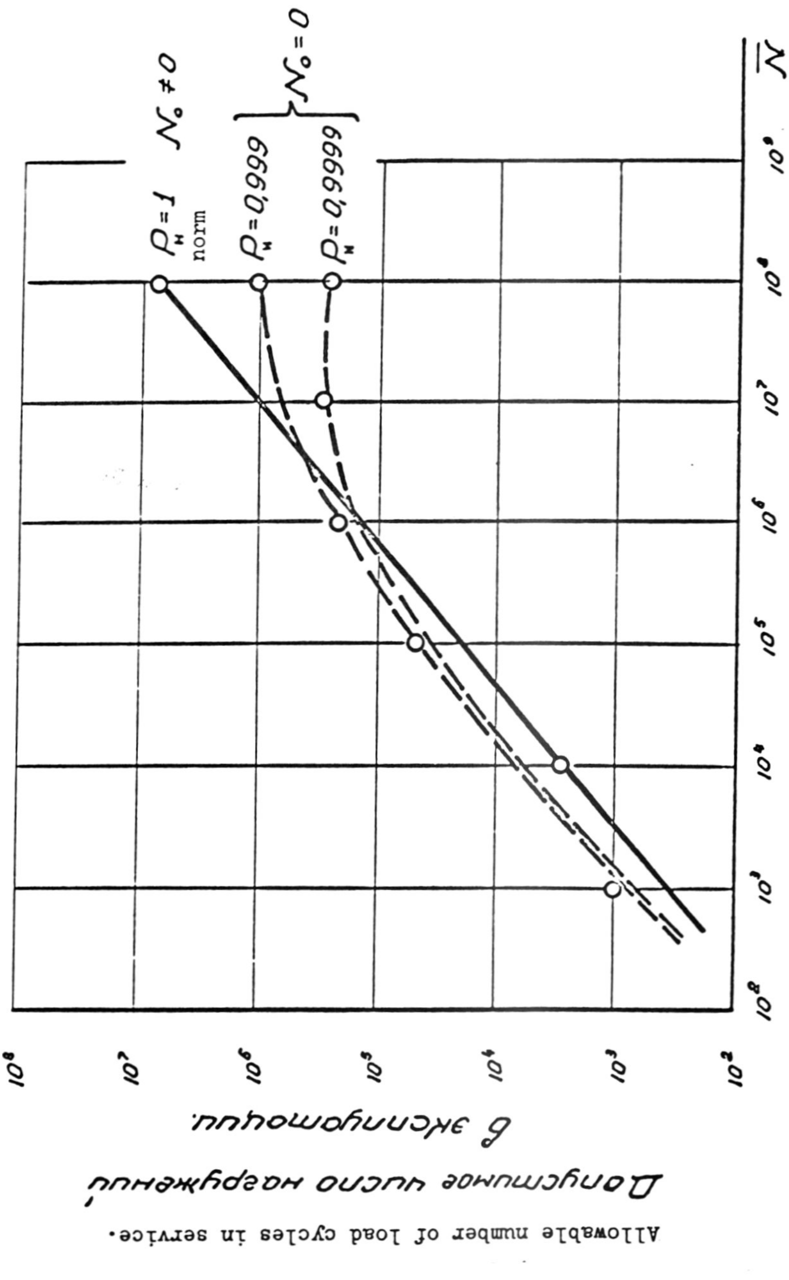

Figure 4. Comparison between allowable service lives for $N_0 = 0$ and $N_0 \neq 0$.

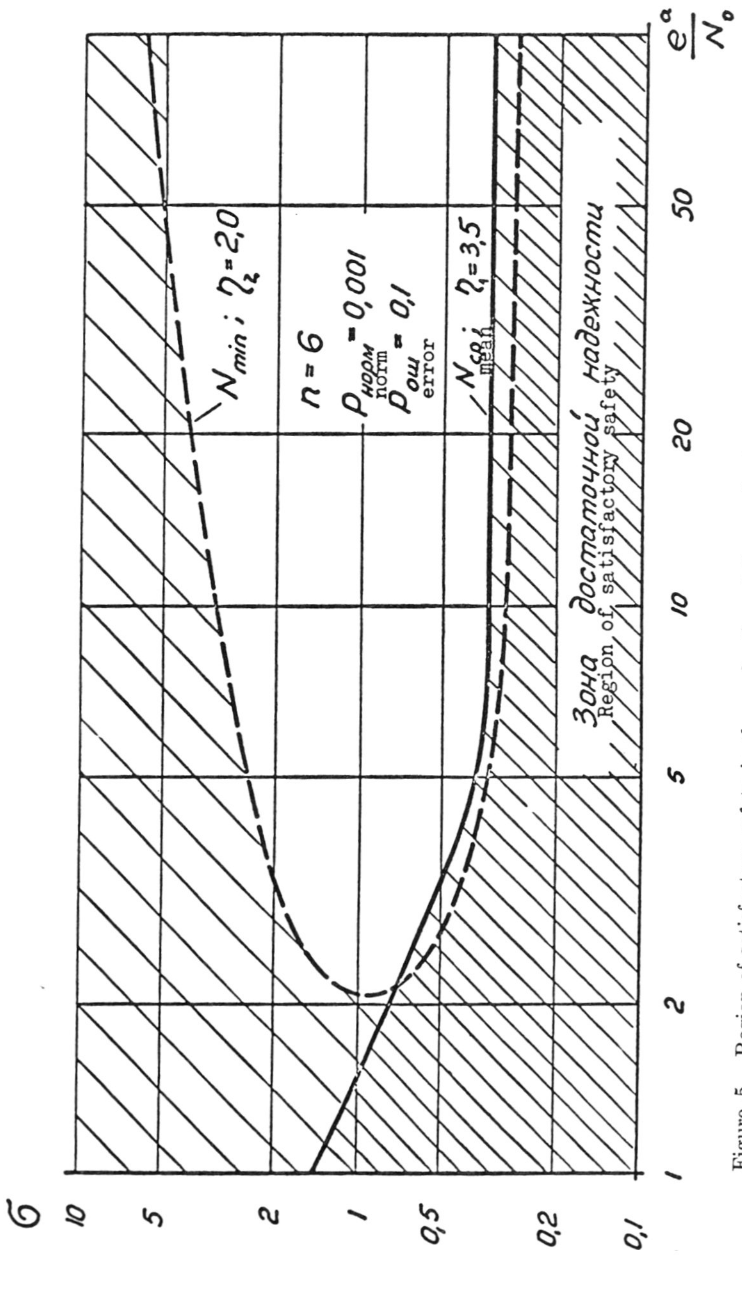

Figure 5. Region of satisfactory safety in the evaluation of the service life for $n = 6$ and $P_{\mathrm{norm}} = 0.35$.

shall be sufficiently reliable for large parameter ranges, the number of test specimens and the factors of safety η must be increased. But even in this case, for example where $\eta = 20$ and $\sigma_{norm} = 0.60$, the parameter range within which the given safety is not secured will be quite large, particularly with evaluation from mean life (Fig. 6). The attainment of a satisfactory safety over the whole of the range of scatter of length of life occurring in practice without an exaggerated increase of the factors of safety is only possible through the calculation of the service life with the determination of the minimum life.

The evaluation of the service life of a structure by calculating with a minimum life is also to be preferred from other points of view. Based on experiments carried out it can be assumed that the magnitude N_0 is to a certain degree more typical for the fatigue strength of materials and structures than is the mean value $\bar{N}$. From statistical analyses of scale effect, for example, it follows that this effect is reduced with increased probability of nonfailure [11]. With zero probability of failure this must completely disappear.

A reduction in the sensitivity in relation to the influence of different factors on the length of life with reduced probability of failure has also been observed in cases of the effect of various types of overloading on the fatigue endurance.

The evaluation of the allowable service life of a structure with respect to fatigue by the calculation of a minimum life is thus appropriate. A method for such evaluation is therefore also necessary. In order to determine minimum life in fatigue it is necessary to study a number of basic theoretical conditions. Among these can be classed in the first instance matters concerning mathematical statistics having to do with the determination of the quantity N_0 experimentally, a method of determining N_0 and possible errors in the same caused by scatter in the test results and the limited sample size of the tests. In addition, a more detailed specification is required of the criterion of the reliability of the structure based on the fatigue conditions. This criterion forms the basis for the determination of safety against fatigue. These questions, which have not yet been treated in the literature on fatigue endurances and service lives, have formed the subject of the present paper.

ACCURACY IN THE DETERMINATION OF THE MINIMUM LIFE

If one considers the length of life dependent upon fatigue as a random variable and the number of test pieces in the fatigue tests is limited (not infinitely large), the value of the minimum life N_0 must, like every other experimentally determined parameter which characterizes the scatter of

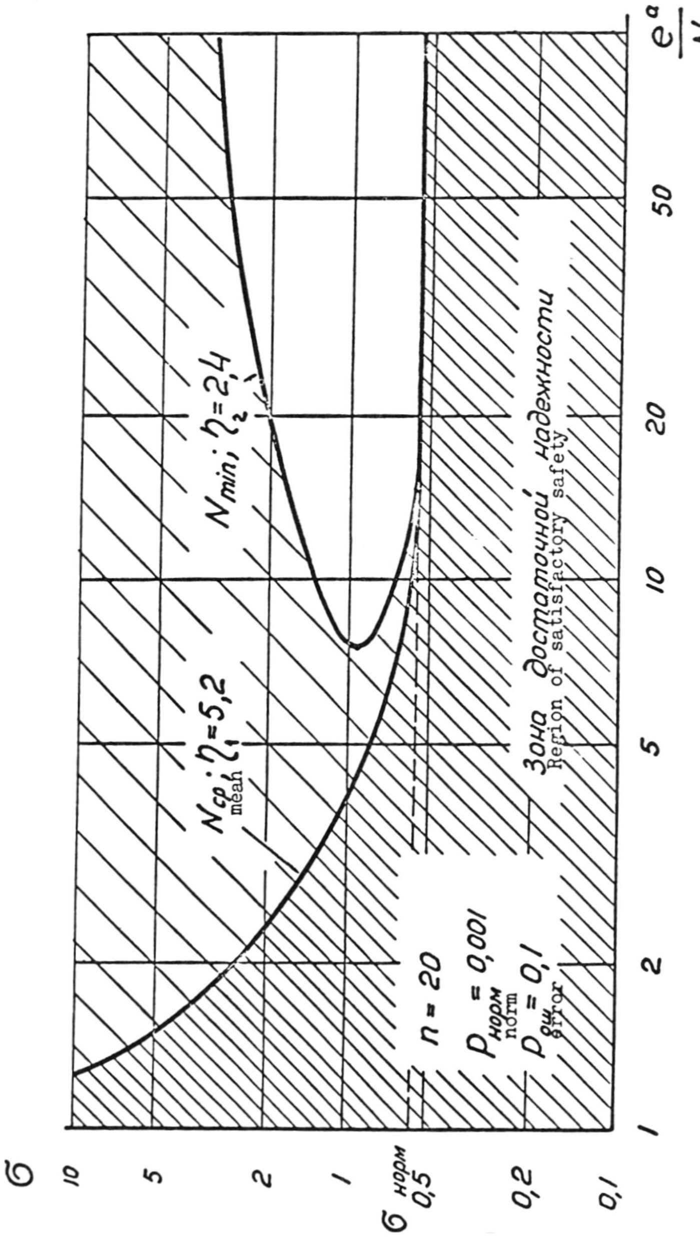

Figure 6. Region of satisfactory safety in the evaluation of the service life for $n = 20$ and $\sigma_{\text{norm}} = 0.6$.

the length of life, be considered to be of random size. The scatter depends above all upon the sample size of the tests and causes a random error in the determination of the minimum life, which can turn out to be of very considerable magnitude and which must be taken into account both in the use of N_0 for the estimation of the allowable service life and also for a correct analysis of the values obtained by experiment. The question of accuracy in the determination of the minimum life has not until now been investigated and therefore existing experimental basic data concerning N_0 must be regarded as guide values.

The spread of the estimated minimum life depends also upon the method that has been used for its determination with the help of experimental values. It is, however, known [12] that how advantageous this method of procedure has been from the point of view of accuracy, the spread for a certain definite number of tests can never fall below a fixed limit. This minimum spread can be determined on the basis of known results from mathematical statistics. It is valuable to know this spread primarily for the determination of the least margin of safety for the service life but also to be able to judge the accuracy of current methods for the estimation of the desired parameter.

We shall solve the problem of determining the minimum limits for the spread of the minimum life on the assumption that the random quantity $\xi = \ln (N - N_0)$, where N is the number of cycles up to failure in fatigue tests and N_0 is the minimum life in number of cycles, has normal distribution with the mean a and the standard deviation σ. The probability density for ξ is in other words:

$$\varphi_\xi = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(\xi-a)^2/2\sigma^2} \tag{1}$$

The spread of the fatigue length of life N_0 is determined in this case by the three parameters $\theta_1 = N_0$, $\theta_2 = a$, $\theta_3 = \sigma$. The quantities N_0, a, and σ are determined from the results of fatigue tests with n test pieces: N_1, N_2, $\ldots$, N_n. The values obtained experimentally (estimations) for these parameters $\bar{\theta}_1 = \bar{N}_0$, $\bar{\theta}_2 = \bar{a}$, and $\bar{\theta}_3 = \bar{\sigma}$ form random quantities whose common spread is above all characterized by the matrix for the second moment of these quantities:

$$|M_{ij}|$$

where

$$M_{ij} = M\{\bar{\theta}_i\bar{\theta}_j\} = \int_{-\infty}^{\infty} \bar{\theta}_i\bar{\theta}_j\varphi_{ij}(\bar{\theta}_{ij}\bar{\theta}_j)\, d\bar{\theta}_i d\bar{\theta}_j$$

$\varphi_{ij}(\bar{\theta}_i\bar{\theta}_j)$ is the function for the common distribution of the estimates $\bar{\theta}_i$ and $\bar{\theta}_j$ for the parameters θ_i and θ_j.

The important quantity, the scatter of a parameter's estimated value, is as is known, its standard deviation σ_{θ_i}, the square of which is equal to the corresponding second moment:

$$\sigma_{\bar{\theta}_i}^2 = M_{ii}$$

The lower limits for the quantities M_{ij} are provided by the matrix coefficients; this matrix is in relation to $|a_{ij}|$ coefficients an inverse quadratic form for so-called ellipsoidal dispersion:

$$\sum_{ij}^{k} a_{ij} (\bar{\theta}_i - \theta_i)(\bar{\theta}_j - \theta_j) = K + 2 \tag{2}$$

K is the number of estimated parameters; in our case $K = 3$.
The coefficients a_{ij} are determined in the following manner:

$$a_{ij} = nM \left\{ \frac{\partial \ln \varphi}{\partial \theta_i} \cdot \frac{\partial \ln \varphi}{\partial \theta_j} \right\} \dots \tag{3}$$

$$M \left\{ \frac{\partial \ln \varphi}{\partial \theta_i} \cdot \frac{\partial \ln \varphi}{\partial \theta_j} \right\} = \int_{-\infty}^{\infty} \frac{\partial [\ln \varphi(\xi)]}{\partial \theta_i} \cdot \frac{\partial [\ln \varphi(\xi)]}{\partial \theta_j} \cdot \varphi(\xi) d\xi \tag{4}$$

After the appropriate calculations have been carried out it is found that the matrix for the coefficients a_{ij} has the following appearance:

$$\begin{bmatrix} \frac{\sigma^2 + 1}{\sigma^2} \cdot e^{-2(a-\sigma^2)}; & \frac{1}{\sigma^2} e^{-(a-\sigma^2/2)}; & -\frac{2}{\sigma^2} e^{-(a-\sigma^2/2)} \\ \frac{1}{\sigma^2}, e^{-(a-\sigma^2/2)}; & \frac{1}{\sigma^2}; & 0 \\ -\frac{2}{\sigma} e^{-(a-\sigma^2/2)}; & 0; & \frac{2}{\sigma^2} \end{bmatrix} \tag{5}$$

The development of the matrix (5) gives the following values for the lower limits of the standard deviation for the estimated values $\tilde{N}_0$, $\tilde{a}$, and $\tilde{\sigma}$:

$$\sigma_{\tilde{N}0} \geqslant \frac{K_1(\sigma)}{\sqrt{n}} e^{a-\sigma^2} \tag{6}$$

$$\sigma_{\tilde{a}} \geqslant K_2(\sigma) \frac{\sigma}{\sqrt{n}} \tag{7}$$

$$\sigma_{\tilde{\sigma}} \geqslant K_3(\sigma) \frac{\sigma}{\sqrt{2n}} \tag{8}$$

The coefficients K_1, K_2, and K_3 are determined in accordance with the formulas:

$$K_1^2 = \frac{\sigma^2}{\sigma^2 + 1 - \frac{1}{2}\left(4\sigma^4 + \sigma^2 + 2\right) e^{-\sigma^2}} \tag{9}$$

$$K_2^2 = \frac{\sigma^2 + 1 - 2\sigma^2 e^{-\sigma^2}}{\sigma^2 + 1 - \frac{1}{2}\left(4\sigma^4 + \sigma^2 + 2\right) e^{-\sigma^2}} \tag{10}$$

$$K_3^2 = \frac{\sigma^2 + 1 - e^{-\sigma^2}}{\sigma^2 + 1 - \frac{1}{2}\left(4\sigma^4 + \sigma^2 + 2\right) e^{-\sigma^2}} \tag{11}$$

The variables $K_1(\sigma)$, $K_2(\sigma)$ and $K_3(\sigma)$ are shown in Fig. 7.

From the formulas quoted above and from Fig. 7 it follows that the lower limit for the standard deviation of $\tilde{N}_0$ does not depend upon the absolute value of the minimum N_0, that it increases with the mean value (expectancy) a and decreases with increased standard deviation σ. The lower limits for the standard deviation of the estimates $\tilde{a}$ and $\tilde{\sigma}$ for $\sigma > 2$ coincide with the exact values of the standard deviation of these estimates for $N_0 = 0$; for $\sigma \leqslant 2$ the quantities σ_σ and σ_σ will be somewhat greater when $N_0 \neq 0$ than at the zero-minimum life. With the aid of the inequality (6) one can make an estimate of the least number of tests that are required to determine the minimum N_0 with a given accuracy. If in the first approximation we use the rule "2^x sigma" and put $K_1 = 1$, we get

$$n_{\min} = \left(\frac{2}{\sigma_{\max}} \cdot \frac{e^{a-\sigma^2}}{N_0}\right)^2 \tag{12}$$

where $\delta_{\max}$ is the allowable maximum relative error in the determination of N_0.

Approximate calculations show that the required number of tests is very large. If, for example, $\delta_{\max} = 0.5$, $\sigma = 1$ and $e^a/N = 10$, which corresponds to an average length of life of the order of 10^9, the least number of test pieces $N_{\min} \approx 200$. For shorter lengths of life combined with reduction of σ and increase of the ratio e^a/N_0 (see Figs. 1 and 2) the number of tests must be even greater.

The choice of the method for the estimation of N_0 will be under such conditions a responsible task. The method used must be effective from the statistical viewpoint, i.e., have the least possible scatter which will as closely as possible agree with the scatter determined in accordance with Eq. (6).

METHODS FOR THE EVALUATION OF THE MINIMUM LIFE

Only one method can be found in the literature for the estimation of the minimum life on the basis of experimental results. This method is based on the smoothing out of the test results in relation to the normal distribution according to the principle of the method of least squares [4]. As is known, the method of least squares gives an estimate both effective and unobjec-

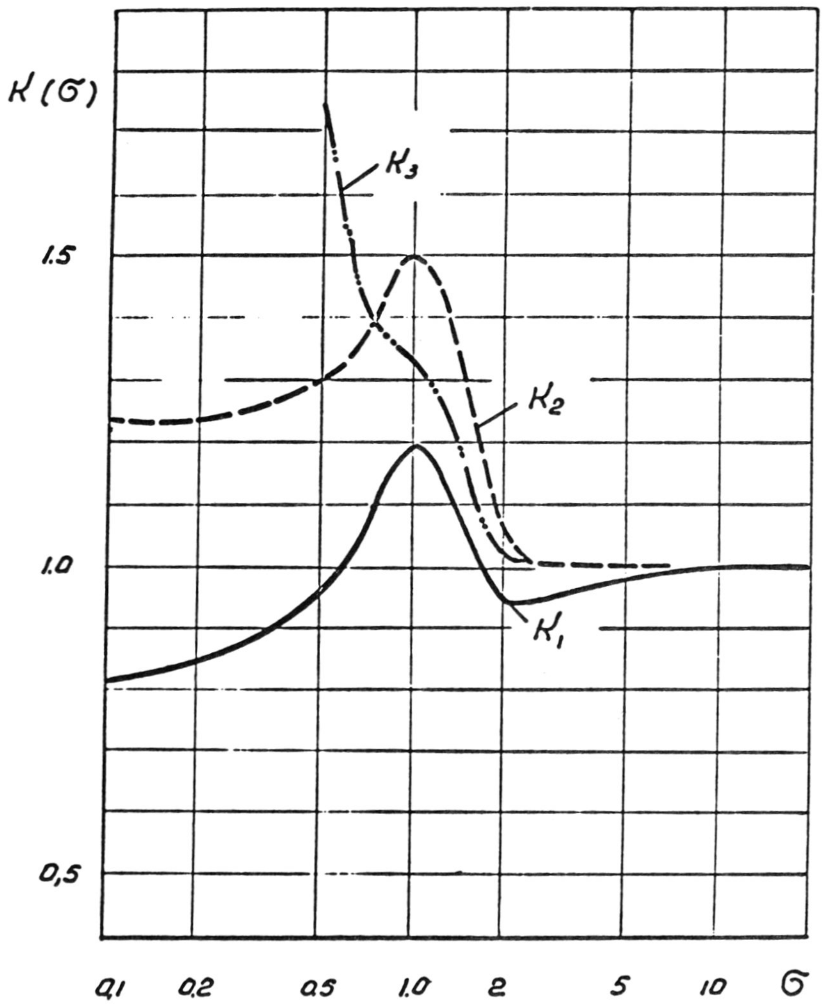

Figure 7. Coefficients K_1, K_2 and K_3 as functions of σ.

tionable* for the case where there exist independent normally distributed quantities with similar standard deviations. With the smoothing out of the terms in a variation series of experimental results these conditions do not exist, and therefore the method of least squares in the form that it has in Eq. (4) demands a study of its effectiveness and even the introduction of certain closer formulations.

In the present work we shall deal with another, more general method for the obtaining of statistical estimates of unknown parameters, namely, the method of greatest probability [12]. The basic assumptions are the same as before, i.e. that the quantity ln $(N - N_0)$ with the parameters a and σ are considered to be normally distributed. The application of the principle of greatest probability gives in the case in question the following equation for estimating $\tilde{N}_0$ for the minimum life at N_0 cycles:

$$\sum_{i=1}^{n} \frac{\ln (N_i - \tilde{N}_0)}{N_i - \tilde{N}_0} = (\tilde{a} - \tilde{\sigma}^2) \sum_{i=1}^{n} \frac{1}{N_i - \tilde{N}_0} \tag{13}$$

In this equation

$$\tilde{a} = \frac{1}{n} \sum_{i=1}^{n} \ln (N_i - \tilde{N}_0) \tag{14}$$

$$\tilde{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} [\ln (N_i - \tilde{N}_0) - \tilde{a}]^2 \tag{15}$$

where $N_1, \ldots, N_n$ are the results from the fatigue tests.

The quantities $\tilde{N}_0$, $\tilde{a}$, and $\tilde{\sigma}$ which are determined from Eqs. (13)–(15) form the estimates of the greatest probability for the parameters N_0, a, and σ. The solution of Eq. (13) can either be done graphically or by known methods of calculation for the solution of algebraic equations. If the number of tests n is very great it is best to make use of an electronic computing machine.

In certain situations the method of greatest probability gives so-called asymptotically effective and normal estimates. If the number n is increased, the standard deviation of the estimates tends towards the minimum limits, but the distribution of the estimates approaches the normal. In order to be able to judge the possibilities of making practical use of this method in the case in question it is necessary to decide how rapidly the estimates obtained approach the effective under the assumption that n increases. This can be estimated, for example, by the use of a method for

* I.e., not containing systematic errors.

statistical tests where the random quantities are modelled in the calculating machine with a given distribution, and where the parameters N_0, a and σ each time are estimated on the basis of the chosen number n. A calculation of this sort has been carried out in which pseudorandom numbers have been employed as random quantities [13]. The calculation was directed towards:

(a) a check of the possibility of making practical use of the estimates in accordance with the maximum method for the determination of the minimum life.
(b) the determination of the actual scatter for the estimates of the minimum life, and the comparison of this scatter with the lower spread limit by means of Eq. (6).
(c) a check of the normality of the distribution of the estimates of $\tilde{N}_0$.

The calculations that have been carried out show that Eq. (13) can be used for the estimation of the minimum life if $n \geqslant 20$. For a smaller number of tests the method in question can give no answer. For the determination of the standard deviation $\sigma_{\tilde{N}_0}$ repeated modellings of the fatigue test results were carried out with given values of the parameters a and σ. On the basis of a series of results obtained for $\tilde{N}_0$, the number of which N_1, has been assigned from 20 to 80, a sample value is determined for the standard deviation of the quantity $\tilde{N}_0$:

$$\bar{S}_{\tilde{N}_0} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n_1}\left(\tilde{N}_{oi} - \frac{1}{n}\sum_{k=1}^{n_1}\tilde{N}_{0k}\right)^2}$$

which at the first approximation has been fixed as the quantity $\sigma_{\tilde{N}_0}$.

The comparison of the values obtained in this way of the standard deviation of the estimate of $\tilde{N}_0$ with their minimum limits indicate that with arbitrary values of N_0 and a in the range $0.5 \leqslant \sigma \leqslant 1.5$ the estimation by means of the maximum method is not far from the effective, beginning at $n = 50$–100 test pieces. Figures 8 and 9 serve as an illustration to this result, where it can be seen how the magnitude $\sigma_{\tilde{N}_0}$ depends upon n, a, and σ.

The distribution of the estimate of $\tilde{N}_0$ approaches the normal starting from 20 tests (see Figs. 10 and 11). The comparison between the distribution of random samples and the theoretical distribution shows satisfactorily good agreement both regarding the character of the distribution and the size and average magnitude of the section and the mean value. Deviations found lie within the permissible limits and can be explained as scatter among the random samples.

The equation of the maximum method (13) can thus be used to determine the dependence of the minimum life upon the number of cycles N_0.

The establishment of possible errors in the determination of N_0 which have been occasioned by random division of the root of Eq. (13) from the actual value, i.e., from the confidence interval of the minimum life, can be done at the first approximation with the aid of Eq. (6), in which case normal distribution for $\tilde{N}_0$ is applied. If n is sufficiently large ($n > 100$–200) the confidence interval for the magnitude N_0, which corresponds to the degree of confidence P_{conf}, is determined in accordance with the formula

$$\tilde{N}_0 - u_\alpha \sigma_{\tilde{N}_0} < N_0 < \tilde{N}_0 + u_\alpha \sigma_{\tilde{N}_0} \tag{16}$$

where u_α, α — = quantile solution of the normal distribution, i.e., that quantity that satisfies the equation

$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2}\, dx; \qquad \alpha = \frac{1 + P_{\text{conf}}}{2}$$

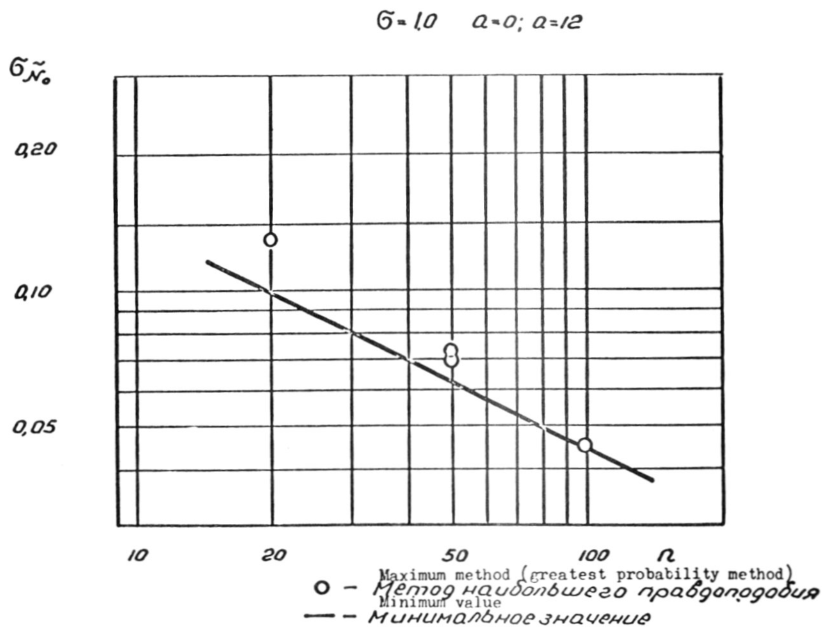

Figure 8. $^\sigma\overline{N}_0$ as a function of the number of fatigue tests.

Instead of a and σ in the initial distribution in Eq. (16) the random values $\tilde{a}$ and $\tilde{\sigma}$ are inserted at the first approximation. With a smaller number of random tests, e.g., for $n = 20$–100, the expression (16) will give far too small an interval since in reality $\tilde{a}$ and $\tilde{\sigma}$ can depart considerably from the actual values of a and σ. In order to obtain a more accurate confidence

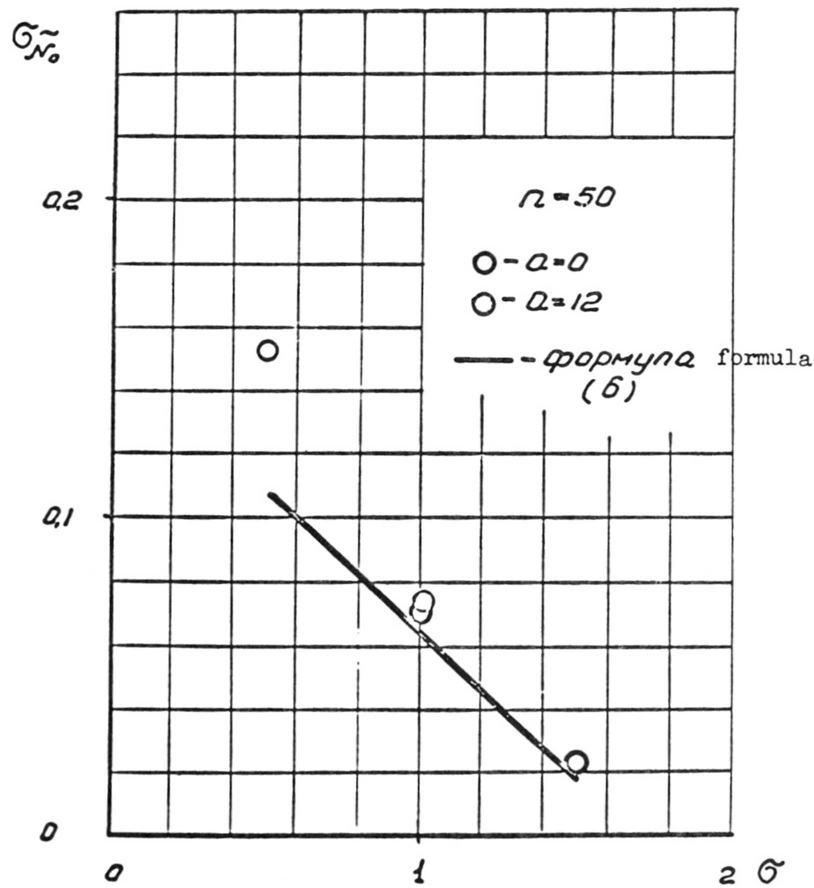

Figure 9. ${}^{\sigma}\overline{N}_0$ as a function of the standard deviation σ.

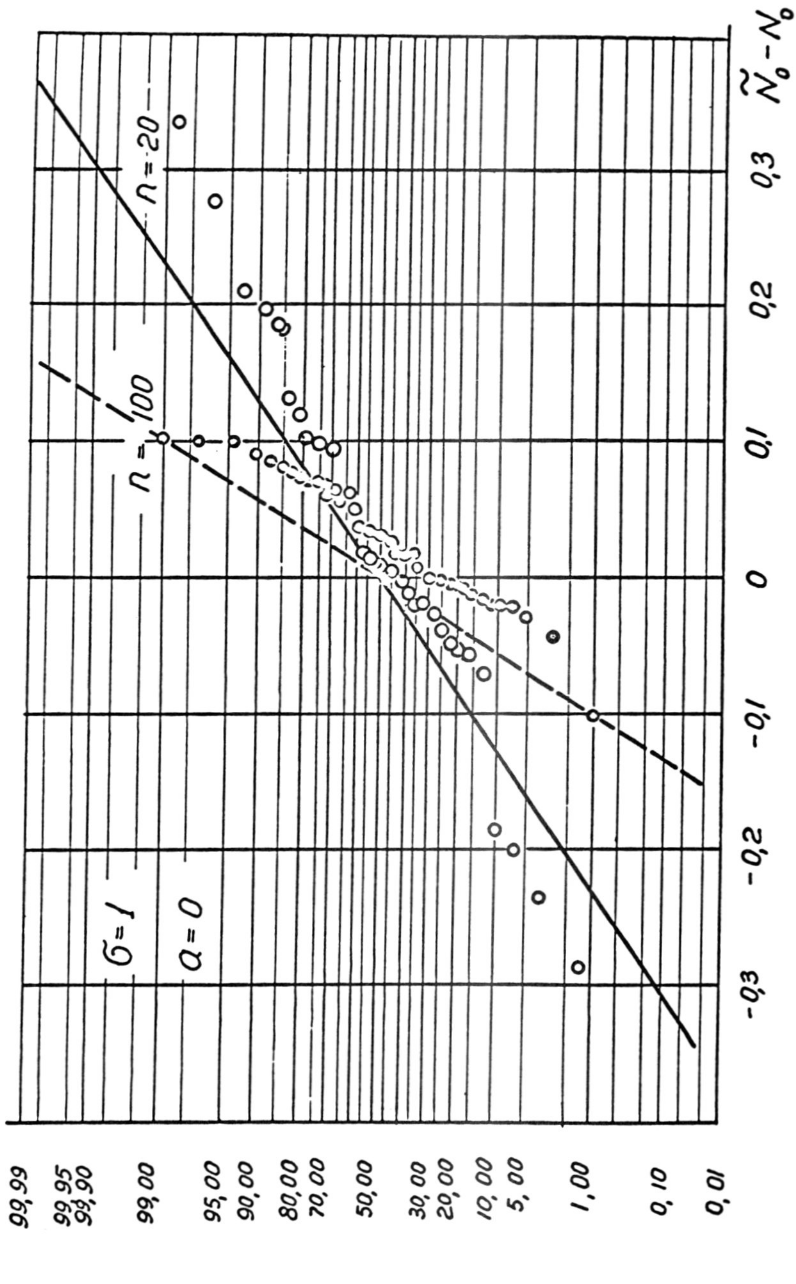

Figure 10. Distribution of the difference $\tilde{N}_0 - N_0$ at $n = 20$ and $n = 100$.

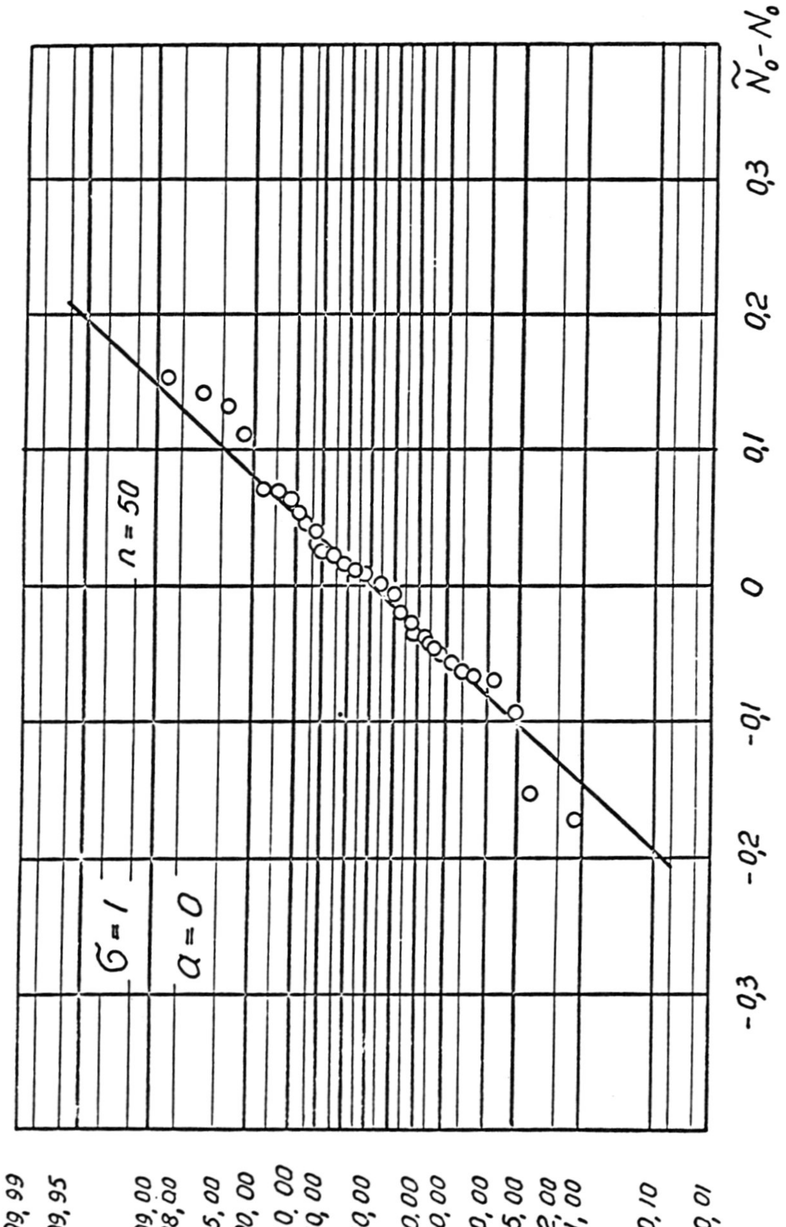

Figure 11. Distribution of the difference $\tilde{N}_0 - N_0$ at $n = 50$.

interval for the minimum life N_0 we make use of the random auxiliary variable κ:

$$\kappa = \frac{\tilde{N}_0 - N_0}{e^{\tilde{a}-\tilde{\sigma}^2}} \tag{17}$$

With the support of formulas (6), (7), and (8) it can be expected that the distribution of the quantity κ will be at any rate slightly dependent on the parameters a and σ. Calculations that have been carried out, a few results from which are given in Fig. 12, indicate that this actually is the situation. The quantity a has no influence at all on the distribution of κ, and the influence of σ is comparatively small in the region where the commonly occurring values of this parameter lie (σ = 0.5–1.5). Figure 13 shows the smoothed out distributions for κ within the limits $0.5 < \sigma < 1.5$ which can be used for the practical determination of the confidence interval of the minimum life. If κ_1 and κ_2 satisfy the conditions

$$\text{Probability } [\kappa < \kappa_1] = 1 - \alpha$$

$$\text{Probability } [\kappa > \kappa_2] = \alpha$$

then the probability that the interval

$$\tilde{N}_0 - \kappa_1 e^{\tilde{a}-\tilde{\sigma}^2} < N_0 < \tilde{N}_0 - \kappa_2 e^{\tilde{a}-\tilde{\sigma}^2} \tag{18}$$

will enclose the actual value of N_0 will be $2\alpha - 1 = P_{\text{conf}}$. The task submitted is thus solved by means of Eq. (18). The quantiites κ_1 and κ_2 can be determined with the aid of the diagram in Fig. 13.

SAFETY CRITERIA IN THE EVALUATION OF THE SERVICE LIFE

In the estimation of the permissible service life two safety criteria, as already established above, are used, namely on the one hand the standardized probability of the collapse of each specimen of a structure during the established service life—P_{norm}, and on the other hand the probability of error in the establishment of the service life—P_{error}, i.e., the probability for the case where in reality the probability of failure $P_{\text{fail}} > P_{\text{norm}}$

$$P_{\text{error}} = \text{Probability } (P_{\text{fail}} < P_{\text{norm}})$$

The service life of the structure is thus a function of two quantities P_{norm} and P_{error}, which offers the possibility of an almost arbitrary choice of one of them. In quite a number of cases, and particularly with log normal

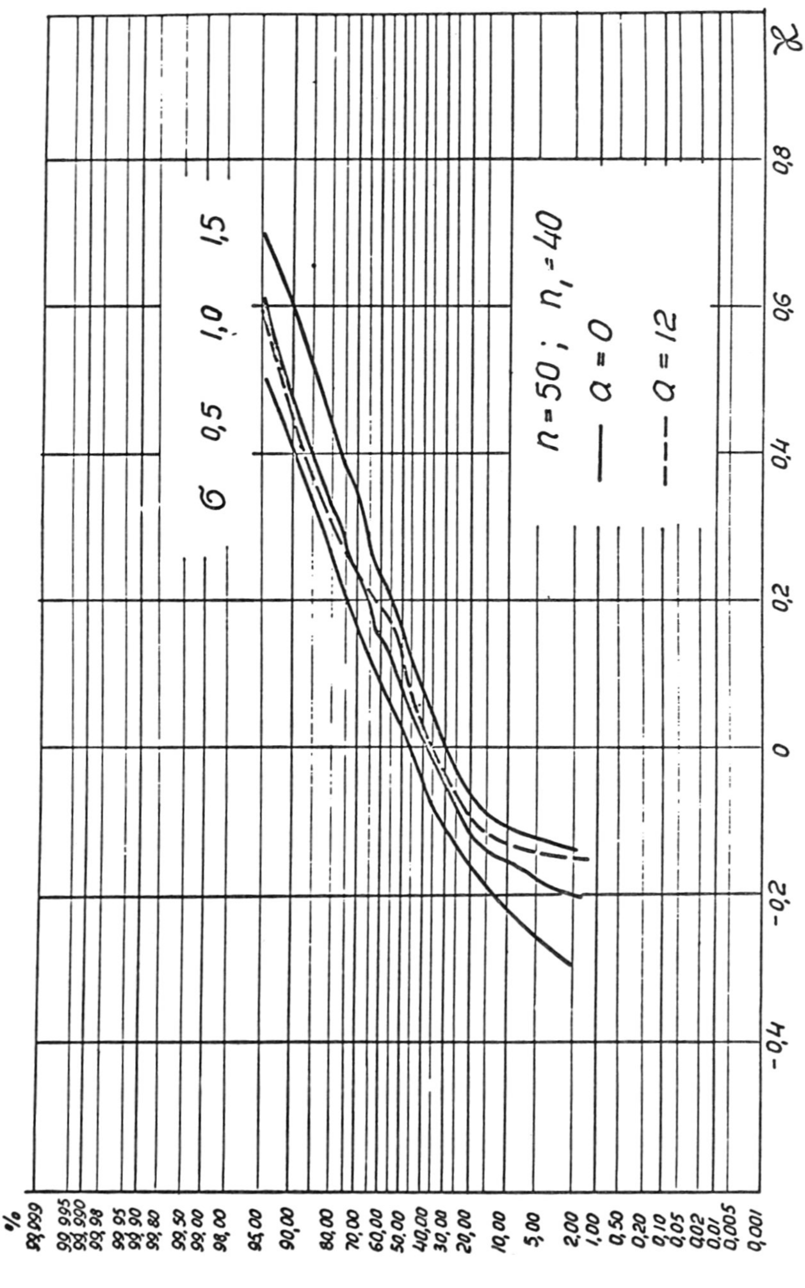

Figure 12. Distribution $\varkappa$ for various values of a and σ.

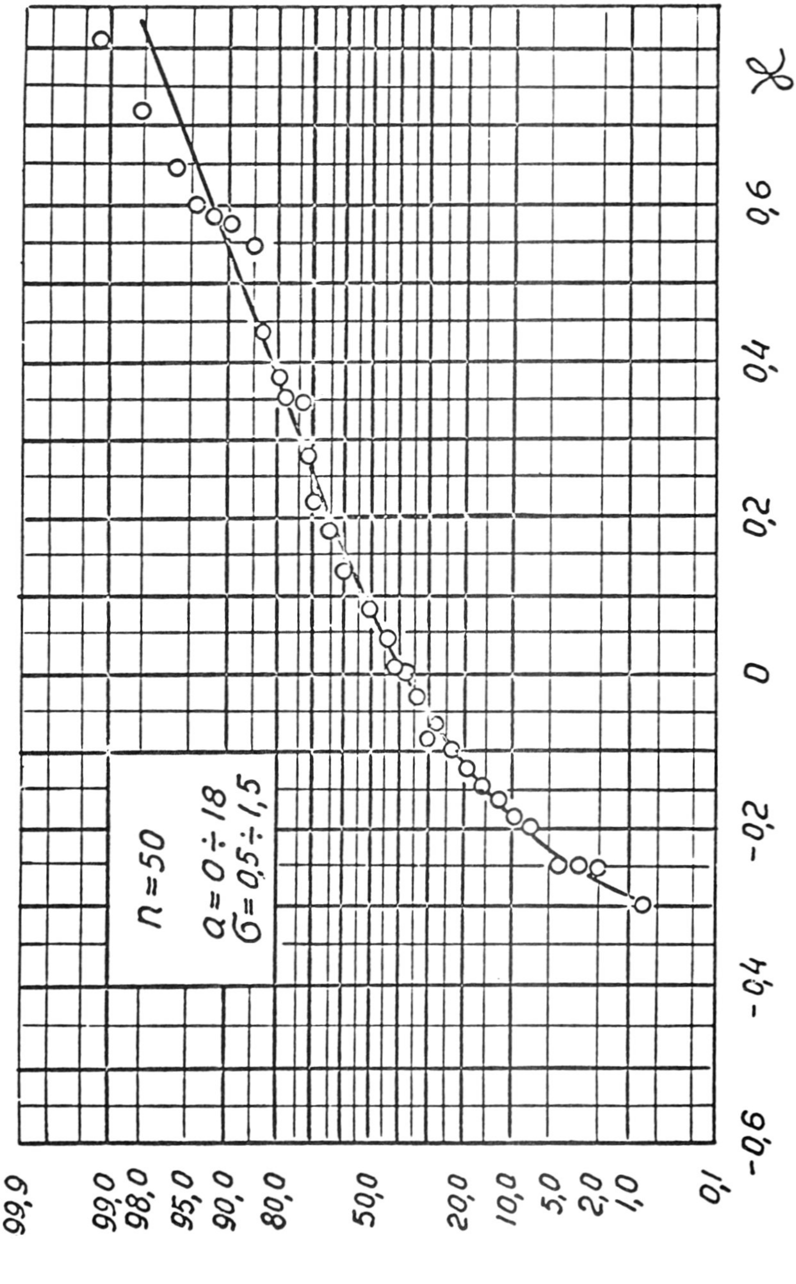

Figure 13. Statistical distribution of $\varkappa$.

distribution of the fatigue life for $N_0 = 0$, this fact does not give rise to any particular difficulties since P_{norm} and P_{error} in each concrete case can be chosen in a suitable way. A different situation arises when there are differences in the character of the distribution, e.g., with strongly deviating values of N_0 or, more exactly, with different ratios of $(a - \ln N_0)/\sigma$. The reliability of the structure with regard to fatigue can in this case with fixed P_{norm} and P_{error} be considerably altered, as will be shown later on. This fact requires the introduction of a more general safety criterion.

Since the principal task in the estimation of the allowable service life is not to permit failure through fatigue during the period of use it is natural to take as a safety criterion the actual probability that failure will not occur. This can be determined in the following manner. Assume that the probability of no-failure is a certain function of the service life expressed in the frequency of the loading cycles.

$$P_{\text{no-failure}} = 1 - F(N).$$

where $F(N) = \int_{-\infty}^{N} \varphi(x)\,dx$ is the integrated distribution of the service life $\varphi\,(N)$ is the probability density for the number of cycles to failure. Assume further that $\varphi_{\text{allowable}}\,(N)$ is the probability density for the allowable working life, i.e., that $\varphi_{\text{allowable}}\,(N)\,dN$ is the probability that the allowable service life of the structure will appear in the interval $N \to N + dN$. Failure in every specimen of the structure is a complex occurrence which when it happens is dependent both on the allowable service life and on the strength of the given specimen. The probability that failure in a sample specimen will not occur during use under the established working life can be determined with the aid of the known formula for the total probability, Eq. (14).

$$P_{H1} = \int_{-\infty}^{+\infty} [1 - F(N)]\,\varphi_g\,(N)\,dN \tag{19}$$

where H_1 = no failures and g describes allowable service life.

As a rule a sufficiently large number of specimens of the given structure can be found in operation, denoted in the following with ν. The probability of failure occurring in all the specimens is:

$$P_{H_\nu} = \int_{-\infty}^{+\infty} [1 - F]^\nu\,\varphi_g\,(N)\,dN \tag{20}$$

The quantity $P_{\text{no-failure } \nu}$ should be suitably taken as the principal criterion for the strength of the structure against fatigue failure since this quantity has a number of properties necessary for this purpose, namely:

1. Expression (20) gives a clear connection between the safety margin in the establishment of the working life and the quantity $P_{\text{no-failure } \nu}$. Assume that $\varphi_{i}(N)_{i}$ [index i = test] is the probability density for the final results of the fatigue tests (mean value of life, minimum life, etc.). Assume further that the service life is established by dividing the test result by the safety factor η:

$$N_{\text{allowable}} = \frac{N_i}{\eta}$$

In this case

$$P_{\text{no-failure } \nu} = \int_{-\infty}^{+\infty} \left[1 - F\left(\frac{N}{\eta}\right)\right]^{\nu} \varphi_{H_i}(N)\, dN \tag{21}$$

which should be shown.

2. The implication of the criterion $P_{\text{no-failure } \nu}$ becomes understandable, from which it becomes comparatively easy to determine the numerical value of the safety investigated. The quantity $P_{\text{no-failure } \nu}$ forms, as indicated above, the probability that the whole of the machine park of the given structure will not be subjected to a single failure during use. This quantity should be chosen so large that the failure of a single machine in the whole machine park will be practically impossible. It would appear that $P_{\text{no-failure } \nu} = 0.99$ can be deemed to be sufficient since thereby is guaranteed on the average one failure in 100 types of sufficiently reliable structures. Depending on the circumstances, the quantity $P_{\text{no-failure } \nu}$ can obviously also take other values.
3. The criterion $P_{\text{no-failure } \nu}$ is to a sufficient degree adaptable with regard to the special properties of the distribution, the method for the establishment of the service life and the number of existing specimens of the structure that are in use.

Let us for example consider the case where the distribution of the length of life $P_{\text{failure}} = F(N)$ is a comparatively slowly growing function with low values of the probability of failure. If the number of test specimens in the fatigue test is sufficiently large which means in consequence that the function $\varphi_{\text{allowable}}$ occupies a sufficiently narrow interval of lengths of life (see Fig. 14) one can put $F(N) \approx P_{\text{norm}}$ and $P_{\text{no-failure } \nu} \approx (1 - P_{\text{norm}})^{\nu}$ $\approx 1 _{-}\nu P_{\text{norm}}$.

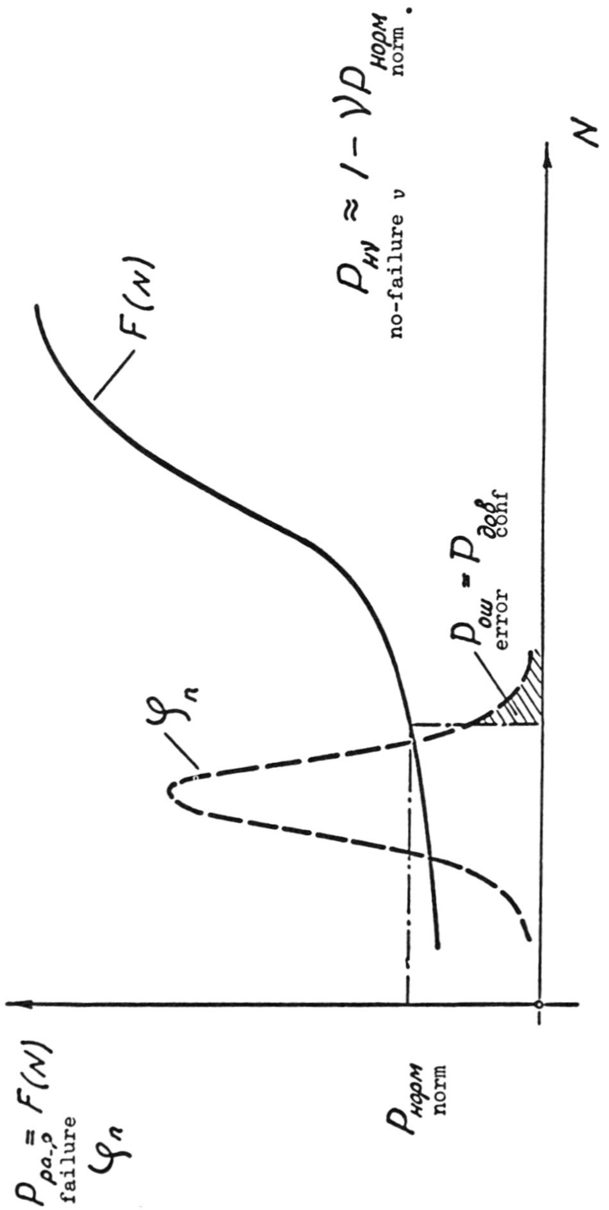

Figure 14. On the determination of the safety $P_{\text{no-failure}}$ with slowly growing probability of failure.

The quantity P_{norm} signifies here the standardized probability of failure of a structure without regard to the factor of safety at the establishment of the service life.

If the distribution of $F(N)$ for small probabilities grows very rapidly and the minimum life is separated from 0 (see Fig. 15) then

$$P_{\text{no-failure }\nu} = \int_{-\infty}^{N_0} [1 - F(N)]^{\nu} \varphi_g(N)dN + \int_{N_0}^{\infty} [1 - F(N)]^{\nu} \varphi_g(N)dN$$

For $N < N_0$, $F(N) = 0$, but for $N > N_0$, $F(N)$ approaches the value 1, or at any rate $[1 - F(N)]^N \approx 0$. Therefore

$$P_{\text{no-failure }\nu} \approx \int_{-\infty}^{N_0} \varphi_g(N)dN \tag{22}$$

The integral of Eq. (22) forms an area which is bounded by the curve $\varphi_g(N)$ and the axis N to the left of the value $N = N_0$. $P_{\text{no-failure }\nu}$ in the given case approaches in other words the value of the confidence probability for the determination of the minimum life.

$$P_{\text{no-failure }\nu} \approx P_{\text{conf}} \tag{23}$$

The properties now enumerated make it possible to regard $P_{\text{no-failure }\nu}$ as an acceptable criterion for the estimation of the service life for different values of the minimum life for the length of life with regard to fatigue.

Especially important in this connection is the relation (23) which shows that the safety with a sufficiently high minimum life, particularly for a large number of ν machines used, is determined by the confidence probability for the determination of the service life, i.e., the minimum life for the length of life with regard to fatigue. Through this conclusion it is possible to establish quantitative values for the factor of safety on the length of life in accordance with the results presented above.

CONCLUSIONS

1. For a number of components and details in aircraft structures, particularly for mechanical details with a certain number of weak points exposed to fatigue loads and with a large machine park, the service life in fatigue must be determined through the calculation of a minimum length of life in fatigue.

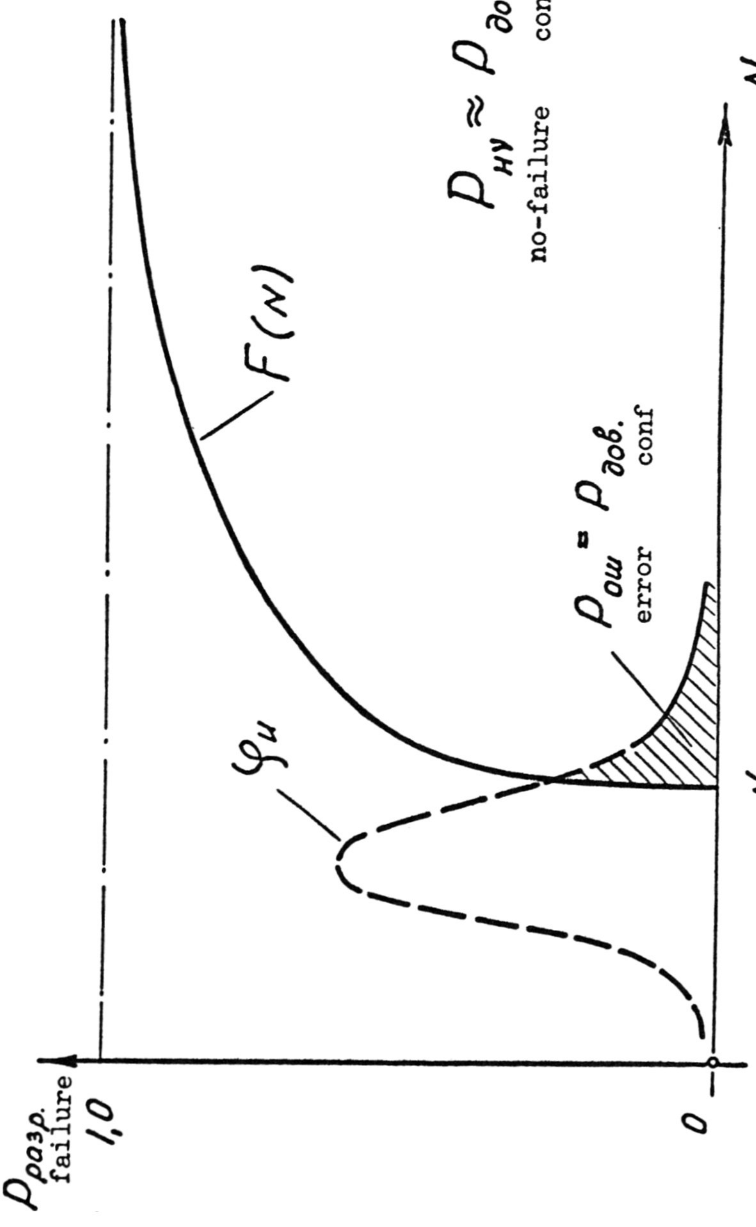

Figure 15. On the determination of the safety $P_{\text{no-failure}}$ for clearly defined sensitivity threshold.

2. In the evaluation of the length of life by means of a minimum value the magnitude of the minimum must be determined by considering the unavoidable errors that occur on account of the statistical scatter of the length of life in fatigue and the limited extent of the fatigue testing.
3. The proposed method for the evaluation of a minimum life with the help of the maximum method has turned out to be suitable. The least number of tests to establish N_0 is at any rate 20. In order to achieve acceptable accuracy this number must be increased ten or twenty times.
4. The calculation of a safe service life by working with a minimum life requires the introduction of a fatigue safety criterion which departs somewhat from those applicable to $N_0 = 0$. One of the possible criteria is the probability of no-failure for the whole machine park, which is determined by considering random errors in the establishment of the allowable service life.

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